29. Metzner, A. B., W. T. Houghton, R. E. Hurd, and C. C. Wolfe, in "Proceedings of International Symposium on Second-Order Effects in Elasticity, Plasticity and Fluid Dynamics," Pergamon Press, New York (1964).
30. Metzner, A. B., W. T. Houghton, R. A. Sailor, and J. L. White, *Trans. Soc. Rheol.*, **5**, 133 (1961).
31. Metzner, A. B., E. L. Carley, and I. K. Park, *Modern Plastics*, **37**, No. 11, 133 (1960).
32. Metzner, A. B., and Malcolm Whitlock, *Trans. Soc. Rheol.*, **2**, 239 (1958).
33. Metzner, A. B., *Ind. Eng. Chem.*, **50**, 1577 (1958).
34. Niller, P. P., and A. C. Pipkin, *Intern. J. Eng. Sci.*, **2**, 305 (1964); *Arch. Rat. Mech. Anal.*, **15**, 1 (1964).
35. Ostwald, Wolfgang, *Kolloid Zeit.*, **36**, 29 (1925).
36. Philippoff, Wladimir, and F. H. Gaskins, *Trans. Soc. Rheol.*, **2**, 263 (1958).
37. Pipkin, A. C., *Rev. Mod. Phys.*, **36**, 1034 (1964).
38. Rajeswari, G. K., and S. L. Rathna, *Z. Angew. Math. Phys.*, **13**, 43 (1962).
39. Reiner, Markus, "Deformation and Flow," Lewis, London (1948).
40. Riggs, L. C., B.Ch.E. thesis, Univ. Delaware, Newark (1956).
41. Rivlin, R. S., in "Phenomenes de Relaxation et de Fluage en Rheologie Non-Lineaire," Editions du Centre National de la Recherche Scientifique," Paris (1961).
42. ———, *J. Rat. Mech. Anal.*, **5**, 179 (1956).
43. ———, and J. L. Ericksen, *ibid.*, **4**, 323 (1955).
44. Sakiadis, B. C., *A.I.Ch.E. J.*, **9**, 706 (1963).
45. ———, *ibid.*, **8**, 317 (1962).
46. Savins, J. G., *Soc. Pet. Eng. J.*, **4**, 203 (1964).
47. Schiller, L., *Z. Angew. Math. Mech.*, **2**, 96 (1922).
48. Schlichting, Herman, "Boundary Layer Theory," 4 ed., McGraw-Hill, New York (1960).
49. ———, *Z. Angew. Math. Mech.*, **14**, 368 (1934).
50. Schowalter, W. R., *A.I.Ch.E. J.*, **6**, 24 (1960).
51. ———, *Chem. Eng. Progr.*, **60**, No. 8, 52 (1964).
52. Shertzer, C. R., and A. B. Metzner, "Proceedings of Fourth International Congress on Rheology," p. 603, Interscience, New York (1965).
53. Tomita, Yukio, *Bull. Japan Soc. Mech. Eng.*, **4**, 77 (1961); *ibid.*, **5**, 443 (1962).
54. Uebler, E. A., and A. B. Metzner, paper presented at Soc. Rheology Pittsburgh Meeting (October, 1964).
55. Wang, Y. L., and P. A. Longwell, *A.I.Ch.E. J.*, **10**, 323 (1964).
56. White, J. L., *J. Appl. Polymer Sci.*, **8**, 1129, 2339 (1964).
57. ———, and A. B. Metzner, *Trans. Soc. Rheol.*, **7**, 295 (1963).
58. ————————, *J. Appl. Poly. Sci.*, **7**, 1867 (1963).
59. ————————, *A.I.Ch.E. J.*, **11**, 324 (1965).
60. White, J. L., and N. Tomita, "Elastomer Processing: Application of Rheological Fundamentals," unpublished manuscript (1965).

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Entrance Region Flow

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The results of rigorous numerical solutions of the general equations of motion are presented for isothermal, laminar, Newtonian flow in a tube entrance region for a uniform entrance velocity. Computed velocity profiles, entrance lengths, and pressure gradients are compared with previous theoretical and experimental results.

In flow from a chamber up to and through a tube, the fluid velocity profile, which at the entrance may vary considerably with conditions, approaches asymptotically that of fully developed flow in the entrance region. The pressure field in a tube entrance region differs from that for fully developed flow in consequence of the kinetic energy changes and the excess viscous dissipation of energy which occur in the establishment of the developed velocity profile. An exact description of the velocity and pressure fields in the entrance region would provide a basis for evaluation of the many approximate solutions of the equations of motion and boundary-layer equations for entrance flow, for analysis and understanding of tube entrance region forced convection energy and mass transport, and for development and evaluation of flow stability theory.

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* In accord with usual practice the entrance length or region is here defined to extend from the beginning of the constant diameter section of the tube to a point where the centerline velocity has increased to 99% of that for fully developed flow.

This study is concerned with Newtonian flow in the entrance region of a tube of circular cross section in which the velocity profile at the tube entrance is approximately flat, such as may occur in viscous flow into a tube with a rounded entrance at relatively high N_{Re}.

In general, solutions of the differential equations of motion for tube entrance flow consist of approximate solutions of restricted forms of the equations of motion, variations in the application of the Prandtl boundary-layer equations, or combinations of these in which a boundary-layer solution valid near the entrance is coupled with a solution of restricted equations of motion which is valid far from the entrance.

By restricting application of the equations of motion in cylindrical coordinates (*1*, p. 85) to conditions such that the flow is independent of time, the radial component of the equations of motion is negligible, any angular motion is negligible (axisymmetric flow), the fluid density and viscosity are constant (isothermal and isobaric flow or temperature and pressure independence), and the flow is independent of any existing body force field, the equations of motion in cylindrical coordinates are reduced to

$$\rho\left(v_r\frac{\partial v_z}{\partial r}+v_z\frac{\partial v_z}{\partial z}\right)=-\frac{\partial P}{\partial z}+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)+\frac{\partial^2 v_z}{\partial z^2}\right] \quad (1)$$

In cylindrical coordinates, the microscopic equation of continuity is

$$\frac{\partial v_r}{\partial r}+\frac{v_r}{r}+\frac{\partial v_z}{\partial z}=0 \quad (2)$$

In obtaining solutions of (1) for flow in a tube entrance region, one or more of the following additional assumptions have been made by previous investigators:

1. Axial molecular transport of momentum is negligible

$$\frac{\partial^2 v_z}{\partial z^2}<<\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) \quad (3)$$

2. The pressure is a function of z and is independent of r

$$P=P(z) \quad (4)$$

3. The velocity at the tube entrance is uniform

$$v_{(\text{entrance})}\neq f(r) \quad (5)$$

4. The effect of radial flow is negligible

$$v_r\frac{\partial v_z}{\partial r}<<v_z\frac{\partial v_z}{\partial z} \quad (6)$$

5.

$$-\frac{1}{\rho}\frac{\partial P}{\partial z}=v_c\frac{\partial v_c}{\partial z} \quad (7)$$

NEW NUMERICAL SOLUTION

In this study (*13*), Equation (1) was solved for the conditions of (3), (4), and (5) by numerical methods with an IBM 7090 digital computer. The entrance region of the tube was considered to comprise N concentric annuli, a radial distance h across, cut by planes perpendicular to the axis into annular segments a distance k long (see Figure 1). Equation (1) in appropriate finite-difference form then was written for each of the $N + 1$ cylindrical boundaries of a set of N annular concentric segments k long. The values of P and the $N + 1$ velocities are known at the entrance to the segments, and the corresponding values at the outlet (end) of the segments were determined by solving the $N + 1$ equations by an iterative process in which the value of P at the end of the segments required to maintain a constant flow rate was determined. A modified Gauss-Seidel method was used to solve the set of $N + 1$ equations for which the matrix of coefficients were of order 200 by 200.

Equation (1), including $\frac{\partial^2 v_z}{\partial z^2}$, is an elliptic partial differential equation and cannot be solved by the initial value technique employed. However, it is of interest that, with this term included, the numerical method employed was operable at $N_{Re} > 200$, which suggests that axial diffusion of momentum is negligible at N_{Re} as low as 200 at least. However, Campbell (*4*) estimates 12% of the drag force to be a consequence of axial diffusion of momentum at $N_{Re} = 600$ and $z^+ \cong 10^{-4}$.

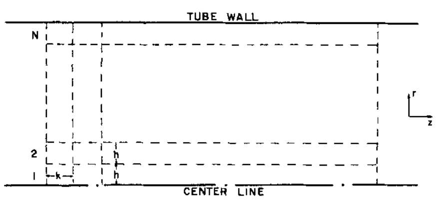

Fig. 1. Division of flow cross section for finite-difference procedure.

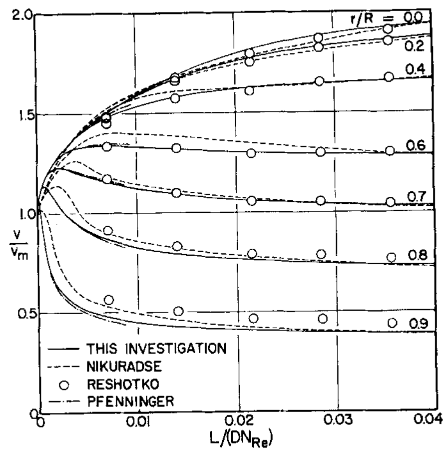

Fig. 2. Reduced experimental velocity and length data of Reshotko, Pfenninger, and Nikuradse compared with computed data of this investigation.

COMPUTED VELOCITY PROFILES COMPARED WITH EXPERIMENTAL DATA

Reduced velocity profiles computed under conditions (3), (4), and (5) and the experimental data of Nikuradse (*15*, p. 27), Pfenninger (*14*), and Reshotko (*17*) are plotted in Figure 2 for comparison. The data of Nikuradse for an unspecified fluid and those of Pfenninger and Roshotko for air are for tubes with rounded entrances. In general, the agreement is fair. However, there are significant differences, especially toward the tube entrance and wall. It should be pointed out that these sets of experimental data are not believed to be sufficiently accurate and/or complete for a rigorous test of the solution of the equations of motion or for flow stability analysis (*5, 21, 26*). It is of interest that, when radial convection of momentum was neglected, the computed velocity profiles were essentially the same, except at the tube centerline, where they were somewhat low.

COMPARISON OF NUMERICAL SOLUTION WITH PREVIOUS THEORETICAL SOLUTIONS

It is instructive to compare the rigorous numerical solution of (1) for conditions (3), (4), and (5) with previous approximate solutions. In order to facilitate comparisons, the computed velocity profiles, together with data from several previous solutions and experimental studies, are plotted in Figures 2, 3, and 4 for r/R of 0.0, 0.6, and 0.9, respectively.

Boussinesq (*3, 9*, p. 306) in a pioneering study solved Equation (1) under the additional conditions of (3), (4), and (5) by using the Poiseuille velocity plus a perturbation correction $f(r, z)$, which satisfied the flow and continuity equations. Data developed by Campbell and Slattery (*4, 5*) from Boussinesq's solution are plotted in Figures 3, 4, and 5, and, as expected, his approximation improves with distance from the entrance. Langhaar (*12*) also solved Equation (1) with the conditions of (3), (4),

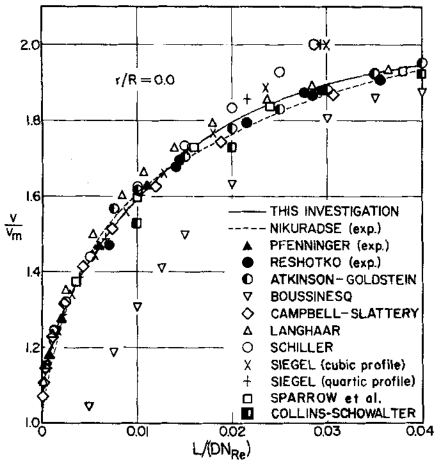

Fig. 3. Representative experimental and computed reduced centerline velocity data plotted vs. reduced length.

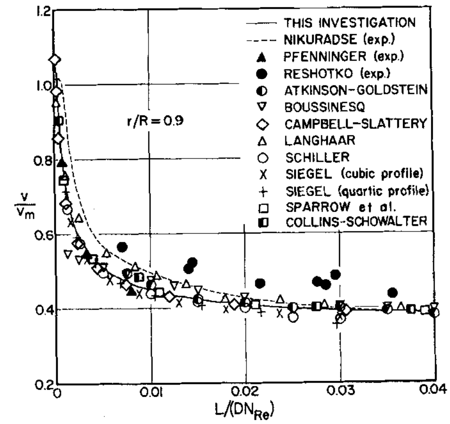

Fig. 5. Representative experimental and computed reduced velocity data at $r/R = 0.9$ plotted vs. reduced length.

and (5), after linearizing Equation (1) by assuming the substantial derivative equal to $\mu[f(z)]^2 v_z$, which is rigorous in the unsheared central core. As indicated in Figures 3, 4, and 5, his solution appears satisfactory at the tube centerline, very near the entrance or far from the entrance. Also, Targ (*25*) linearized Equation (1) by assuming the substantial derivative equal to $v_m\, \partial u/\partial z$ and replaced $\frac{1}{\rho}\frac{\partial P}{\partial z}$ with $(2\nu/R)\, \partial u/\partial r$ to obtain a solution. More recently, Sparrow et al. (*27*) employed the following linearized form of Equation (1).

$$\epsilon(z)\, v_m(\partial v_z/\partial z) = \Lambda(z) + \nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)\right] \quad (8)$$

which is similar to but embodies a more general linearization of the inertial terms than that used by Targ, and matched pressure gradients based on momentum and energy considerations to obtain a solution. Velocity profiles by Sparrow et al. (*27*) are nearly identical with those computed by the presently reported numerical solution (see Figures 3 and 5). By Targ's solution, which neglects the contribution of momentum change to pressure gradient, velocity profiles develop more slowly near the entrance than is predicted by our numerical solution.

COMPARISON OF NUMERICAL SOLUTIONS WITH BOUNDARY-LAYER SOLUTIONS

The basic conditions of boundary-layer theory do not exist in tube entrance region flow and are closely approximated only near the tube entrance, where the boundary layer is relatively thin. Nevertheless, many useful boundary-layer solutions have been achieved. In most of these solutions, Equation (1) under restrictions (3), (4), and (5) is integrated across the boundary layer to obtain a momentum integral equation (an approximation) into which a suitable boundary-layer velocity profile $v_{(r,z)}$ is inserted. A form of the Bernoulli equation

$$\Delta\frac{1}{2}v^2 + \frac{1}{\rho}\Delta P = 0 \quad (9)$$

is applied to an unsheared (assumed) core. The restrictions of Equations (3), (4), (5), and (6) are implicit in Equation (9). Thus Schiller (*19*) employed restrictions (3), (4), (5), and (6) and a parabolic velocity profile in his momentum equation for the developing boundary layer and applied (9) to the unsheared core.

Schiller's procedure was repeated essentially by Siegel (*22, 24*) for modified cubic and modified quartic boundary-layer velocity profiles and by Bogue (*2*) for cubic velocity profiles and power law flow. The equation for power law flow

$$\tau = m(\dot{S})^n \quad (10)$$

represents Newtonian flow when $n = 1.0$. Bogue included the radial momentum term $v_r \frac{du}{dr}$ in the von Kármán integral method (*20*, p. 137). Nevertheless, results obtained by Bogue's solution for $n = 1$ are essentially the same as those for Schiller's solution. As shown in Figures 3, 4, and 5, velocity profiles from these integral method solutions vary little with the assumed boundary-layer velocity profile, are best near the tube entrance, and deviate considerably far from the entrance, especially at the centerline, where boundary-layer theory does not apply. Tomita achieved similar results for power law flow and $n = 1$ by boundary-layer solutions with Schiller's method (*29*) and a variational method (*30*). Recently, to account for viscous dissipation of energy in the boundary layer, Campbell and Slattery (*5*) modified Schiller's method in that they applied a form of (9) plus a viscous dissipation of energy term F (*1*, p 212) to the entire flow in computing pressures, whereas Schiller et al. applied Equation (9) only to the unsheared core. Their solution is much improved over Schiller's, especially at greater distances from the entrance, where an increasingly large part of the fluid is in boundary-layer shear and viscous dissipation of energy is increasingly important.

Tatsumi (*28*) simplified Equation (1) by assuming (3), (5), (7), an undeformed central core, and other conditions employed in boundary-layer theory and by assuming a stream function such that velocity profiles are

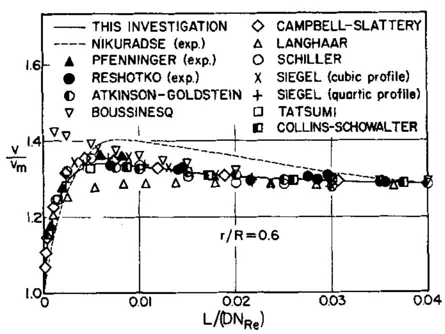

Fig. 4. Representative experimental and computed reduced velocity data at $r/R = 0.6$ plotted vs. reduced length.

almost similar. Velocity profile data by his boundary-layer model solution are in good agreement with those of Pfenninger and with those by our numerical solution very near the tube entrance. However, deviations become significant as z^+ approaches 0.005, where his analysis indicates that the undeformed core assumption is invalid. Tatsumi's method is a variation of the "downstream" analysis employed for flow in a channel by Schlichting (*20*, p. 169).

Atkinson and Goldstein (9, pp. 304-308) also employed a variation of Schlichting's method (*20*, p. 169). They simplified (1) by (3), (4), (5), and (7) and assumed a stream function

$$\psi = -R^2 v_m \sum_{i=1} [(4z^+)^{1/2}]^i f_i(\eta) \qquad (11)$$

to obtain a boundary-layer model solution which is believed to be accurate at $z^+ \lesseqgtr 0.0006$ (*9*, p. 306). As suggested by Schlichting, this solution was then coupled at $z^+ = 0.0006$ with a Boussinesq type of solution which is valid far from the entrance. Their coupled relationship yields velocity profile data which agree with Nikuradse's centerline data, with Reshotko's data at $r/R = 0.6$, and with our computed data at $z^+ \gtreqless 0.01$ except near the centerline. Similarly, Punnis (*16*) joined a downstream boundary-layer solution to a Boussinesq type of solution at $z^+ = 0.0004$ to obtain a solution which appears to be inferior to that of Atkinson and Goldstein. Recently, Collins and Schowalter (*7*) analyzed power law flow by a procedure analogous to that employed by Atkinson and Goldstein and computed velocity profile data for Newtonian flow ($n = 1$) which agree with those of Atkinson and Goldstein except at the centerline, where the deviation from data by our numerical solution is greater.

COMPUTED AND EXPERIMENTAL PRESSURE GRADIENTS

Entrance region pressure gradients commonly are reported as the excess C of Equation (12) over that for Poiseuille flow

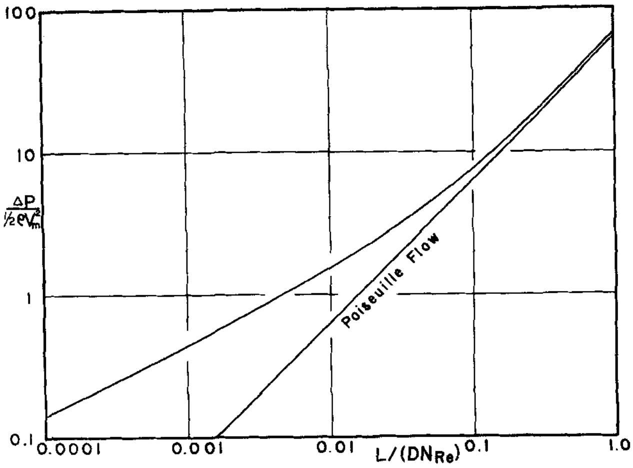

Fig. 6. Computed entrance region pressure gradient data from this study plotted vs. reduced length.

$$\frac{2(P_o - P_e)}{\rho v_m^2} = \frac{64\,L}{D\,N_{Re}} + C \qquad (12)$$

Included in C is the contribution for the change from a uniform to a parabolic velocity profile and the entrance friction loss in excess of that for Poiseuille flow, $C - 1$. Computed and experimental values of C, as well as reduced entrance region lengths, are tabulated in Table 1. In general, results from our numerical solution agree with previously reported computed and experimental data and we believe them to be the most reliable. It is of interest to note that the inclusion of the radial convection of momentum in the numerical solution significantly increased the pressure loss, whereas the pressure loss by Schiller's boundary-layer solution, in which radial convection is neglected, is identical with that from Bogue's repetition of Schiller's solution, in which the radial term was retained.

Computed reduced pressure gradient data are presented in Figure 5 as a function of the reduced length. These data, together with data from previous experimental and analytical studies, are also presented in Figure 6 in a form which facilitates acute discrimination. The data computed by means of the several theoretical analyses differ considerably. However, data by our numerical solution are relatively close to the mean of the experimental data. Furthermore, in comparison with Equation (13)

TABLE 1. VALUES OF C [EQUATION (12)] AND REDUCED ENTRANCE LENGTHS

		C	$z/(DN_{Re})$
Experimental			
Dorsey	(*8*)	1.08, 1.00	—
Knibbs	(*11*)	1.27 ± 8%	—
Nikuradse	(*15*)	1.32	0.0625
Rieman	(*18*)	1.248 ± 1%	—
Schiller	(*19*)	1.32 ± 10%	—
Weltmann and Keller	(*31*)	1.2 ± 10%	—
Theoretical			
This investigation [Equation (1) with radial term]		1.274	0.0555
This investigation [Equation (1) without radial term]		1.015	—
Atkinson and Goldstein	(*9*)	1.41	0.065
Bogue	(*2*)	1.16	0.0288
Boussinesq	(*3*)	1.24	0.065
Campbell and Slattery	(*5*)	1.18	0.0675
Collins and Schowalter	(*7*)	1.33	0.061
Langhaar	(*12*)	1.28	0.0575
Schiller	(*19*)	1.16	0.0288
Siegel (cubic profile)	(*24*)	1.08	0.0300
Siegel (quartic profile)	(*24*)	1.106	0.0296
Sparrow, Lin, and Lundgren	(*27*)	1.24	—
Tomita	(*30*)	1.22	0.0505

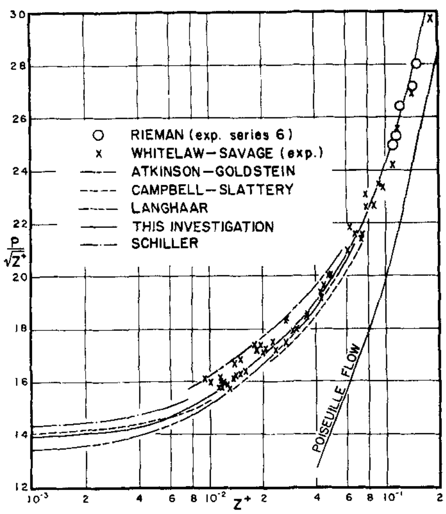

Fig. 7. Comparison of computed and experimental entrance region pressure gradient data from several investigators.

$$\frac{P_e - P_o}{\frac{1}{2}\rho v_m^2} = \frac{13.74}{\sqrt{z^+}} \tag{13}$$

which represents the mean of the high N_{Re} data of Kline and Shapiro for $10^{-6} < z^+ < 10^{-3}$ (*10, 22*), our computed data agree almost exactly in the lower range and are only about 2% higher at $z^+ = 10^{-3}$.

CONCLUSIONS

Although the available experimental data are inadequate for a very rigorous test of our numerical solution of the equations of motion for Newtonian flow in a tube entrance region, the agreement of computed with experimental velocity profile, entrance length, and pressure gradient data indicates that a close approximation to reality for the prescribed conditions has been achieved. The results are expected to be useful in viscometry, in flow stability studies, and in the development of theory concerned with entrance region flow and energy and mass transport in the case of Newtonian flow.

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NOTATION

A = area L^2
C = entrance region excess pressure gradient, dimensionless
D = diameter, L
F = rate of viscous dissipation of energy per unit of mass, $L^2\theta^{-3}$
h = radial dimension of finite-difference grid, L
k = axial dimension of finite-difference grid, L
L = length, L
M = mass
m = constant in Ostwald de Waal power law equation, $M\theta^{n-2}L^{-1}$
N = number of annular segments in finite-difference grid
n = constant in Ostwald de Waal power law equation, dimensionless
P = pressure, $ML^{-1}\theta^{-2}$
R = radius of tube, L
r = radial distance, cylindrical coordinate, L
$\dot{S}$ = shearing rate or rate of strain, θ^{-1}
T = temperature
v = velocity, $L\theta^{-1}$
z = axial distance along tube, cylindical coordinate, L

Subscripts

c = centerline property
e = property at tube entrance
m = mass average
o = property at tube or section exit
r = radial property or direction
z = axial or coordinate property or direction

Superscript

$+$ = normalized or reduced property

Greek Letters

θ = time
ν = μ/ρ
τ = shearing stress or momentum flux $ML^{-1}\theta^{-2}$
μ = viscosity or consistency, $ML^{-1}\theta^{-1}$
ρ = density, ML^{-3}
ψ = stream function
$\epsilon(x)$ = $f(x)$ which weights v
$\Lambda(x)$ = includes dP/dx plus residual inertia terms

Dimensionless Groups

N_{Re} = Reynolds number, $Dv\rho/\mu$
z^+ = reduced axial distance, $z/(DN_{Re})$ or $L/(DN_{Re})$, dimensionless
η = $(1 - r^2/R^2)/4(4z^+)^{1/2}$, dimensionless

Mathematical Symbols

f = function of
Δ = finite increment

LITERATURE CITED

1. Bird, R. B., W. E. Stewart, and E. N. Lightfoot, "Transport Phenomena," Wiley, New York (1960).
2. Bogue, D. C., *Ind. Eng. Chem.*, **51**, 894 (1959).
3. Boussinesq, J., *Compt. Rend.*, **113**, 9-15, 49-51 (1891).
4. Campbell, W. D., M.S. thesis, Northwestern Univ., Evanston, Ill. (1960).
5. ———, and John C. Slattery, *J. Basic Eng.*, 41-46 (March, 1963).
6. Collins, Morton, Ph.D. thesis, Princeton Univ., N. J. (1962).
7. ———, and W. R. Schowalter, *A.I.Ch.E. J.*, **9**, 804-809 (1963).
8. Dorsey, N. E., *Phys. Rev.*, Pt. 2, **28**, 833 (1926).
9. Goldstein, Sydney, "Modern Developments in Fluid Dynamics," Vol. 1, Clarendon Press, Oxford, England (1938).
10. Kline, S. J., Sc.D. thesis, Massachusetts Inst. Technol., Cambridge (1952).
11. Knibbs, G. H., *Proc. Roy. Soc. N. S. Wales*, **29**, 77 (1895).
12. Langhaar, H. L., *J. Appl. Mech.*, **9**; *Trans. Am. Soc. Mech. Engrs.*, **64**, A-55-58 (1942).
13. Lemmon, Hal E., Ph.D. thesis, Univ. Utah, Salt Lake City (1963).
14. Pfenninger, W., "Boundary Layer and Flow Control," G. V. Lackmann, ed., Vol. 2, pp. 970-980, Pergamon, New York (1961).
15. Prandtl, Ludwig, and O. J. Tietjens, "Applied Hydro- and Aeromechanics," McGraw-Hill, New York (1934).
16. Punnis, B., thesis, Göttingen (1947).
17. Reshotko, E., *Jet Propulsion Lab. Progr. Rept. No. 20-364*, California Inst. Technol. (October 24, 1958).
18. Rieman, W. J., *J. Am. Chem. Soc.*, **50**, 46-55 (1928).
19. Schiller, Ludwig, *Z. Angew. Math. Mech.*, **2**, 96-106 (1922); *Phys. Z.*, **23**, 14 (1922); *Forshungsarbeiten V.D.I.*, **248** (1922).
20. Schlichting, Hermann, "Boundary Layer Theory," 4 ed., McGraw-Hill, New York (1960).
21. Shapiro, A. H., *J. Basic Eng.*, 45 (March, 1963).
22. ———, R. Siegel, and S. J. Kline, "Proc. 2 U. S. Natl. Congr. Appl. Mech.," pp. 733-741, Am. Soc. Mech. Engrs., New York (1954).
23. Sibahayasi, M., and E. Sugino, in "Proc. 4th Japan Natl. Congr. Appl. Mech.," pp. 297-298 (1954).
24. Siegel, Robert, Sc.D. thesis, Massachusetts Inst. Technol., Cambridge (1953).
25. Slezkin, N. A., "Dynamics of Incompressible Viscous Fluids," p. 357, Gos. Techteor. Izdat., Moscow (1955).
26. Smith, A. M. O., *J. Fluid Mech.*, **7**, 565-576 (1960).
27. Sparrow, E. M., S. H. Lin, and T. S. Lundgren, *Phys. Fluids*, **7**, 338-347 (1964).
28. Tatsumi, Tomomasa, *J. Phys. Soc. Japan*, **7**, 489-495 (1952).
29. Tomita, Yukio, *Soc. Chem. Engrs. Japan*, **23**, 525-529 (1959).
30. ———, *Bull. Soc. Mech. Engrs.*, **4**, No. 13, 77-86 (1961).
31. Weltmann, R. N., and T. A. Keller, *Natl. Advisory Comm. Aeronaut. Tech. Note* 3889 (1957).

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